An Efficient Implementation of Berenger's Perfectly Matched Layer (PML) for Finite-Difference Time-Domain Mesh Truncation

Jonathon C. Veihl, *Student Member, IEEE,* and Raj Mittra, *Life Fellow, IEEE*

Abstract— **In this letter, an efficient three-dimensional (3-D) implementation of the perfectly matched layer (PML) type of absorbing medium is presented. The technique combines a new eight-unknown time domain formulation in regions in which there is only one nonzero conductivity with the original 12-unknown formulation in the edge and corner regions where nonzero conductivities overlap. Numerical examples of radiation and guided wave problems are included to demonstrate that the modified formulation provides accuracy comparable to the original split field formulation while substantially reducing the memory and CPU requirements of the PML regions.**

I. Introduction

Berenger [1] recently introduced a novel concept for designing a "Perfectly Matched Layer" (PML) that provides reflectionless absorption of electromagnetic waves independent of frequency or angle of incidence. It has been demonstrated that PML provides unmatched performance in the ability to provide reflectionless mesh truncation for three-dimensional (3-D) finite-difference time-domain simulations [2], [3]. The ability to absorb outgoing waves is provided by the additional degrees of freedom introduced by a split field formulation with anisotropic material properties. However, this improved performance comes at a price: to effectively absorb the energy, a PML region of sufficient depth must be added onto the computational domain. Even though the PML medium may be placed much closer to the scatterer than possible with traditional ABC's, the addition of the PML regions (which have twice the number of unknowns as the traditional Yee scheme) may result in increased storage and CPU requirements. Previously, a modified formulation was presented that yields Maxwell's equations with an additional dependent source term [4], [5]. In this letter, additional details and numerical verification of the accuracy of the new technique are presented for the time domain. For wall regions, only eight unknowns are required. In the much smaller edge and corner regions, the original 12-equation formulation is used and an interface condition is applied.

II. Perfectly Matched Layer—Split Equations

The defining property of PML absorber that, in theory, provides reflectionless absorption independent of frequency or angle is

$$\frac{\sigma_i}{\varepsilon} = \frac{\sigma_i^*}{\mu}, \quad i = x, y, z. \tag{1}$$

The original 3-D extension of Berenger's PML concept results in 12 equations in 12 split field unknowns [2] of which two representative equations are reproduced below

$$\mu_0 \frac{\partial H_{xy}}{\partial t} + \sigma_y^* H_{xy} = -\frac{\partial (E_{zx} + E_{zy})}{\partial y} \tag{2a}$$

$$\mu_0 \frac{\partial H_{xz}}{\partial t} + \sigma_z^* H_{xz} = \frac{\partial (E_{yx} + E_{yz})}{\partial z}. \tag{2b}$$

The original PML implementation employed exponential differencing in time due to the rapid decay of the fields. However at a given point in space within the PML, the field behavior is not rapidly decaying in time. The field does decay rapidly in space; however, exponential spatial differencing was not used. Use of standard central differencing for a lossy medium results in equivalent accuracy levels. Since both methods have the same computational cost, there is no computational advantage to either method. However, a test case has been contrived that suggests exponential differencing to be more limiting in the range of allowed conductivity values than central differencing. Simulation of a PML-filled cavity with constant $\sigma = 0.012$ leads to instability when exponential differencing is employed; however, instability was not observed with central differencing as shown in Fig. 1. Central differencing is used in the technique presented below.

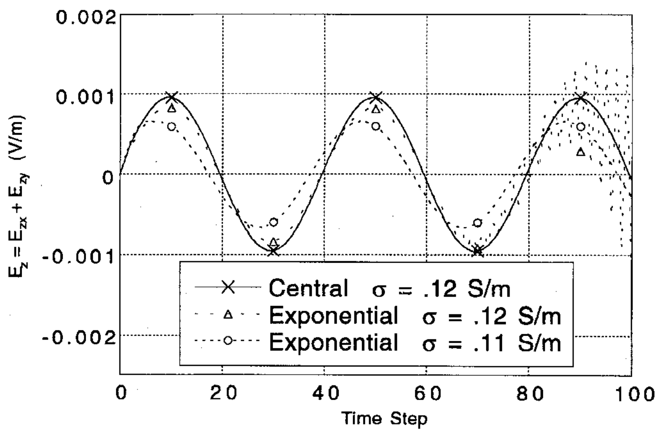

Fig. 1. E_z at position (12, 12, 12) due to a 1.0-GHz sinusoidal source at center of a $20 \times 20 \times 21$ cell PEC cavity filled with PML medium of constant conductivity; $\Delta x = 15$ mm, $\Delta t = 25$ ps.

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The authors are with the Electromagnetic Communication Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA.
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III. Modified Unsplit Time Domain PML

The unsplit formulation in the frequency domain is used as a starting point [6]. For attenuation in the z-direction, the relevant equations for the magnetic field components are

$$j\omega\mu_0\tilde{H}_x = jk_y\tilde{E}_z - \frac{jk_z}{1 - j\frac{\sigma_z^*}{\omega\mu_0}}\tilde{E}_y \tag{3a}$$

$$j\omega\mu_0\tilde{H}_y = \frac{jk_z}{1 - j\frac{\sigma_z^*}{\omega\mu_0}}\tilde{E}_x - jk_x\tilde{E}_z \tag{3b}$$

$$j\omega\mu_0\tilde{H}_z = jk_x\tilde{E}_y - jk_y\tilde{E}_x. \tag{3c}$$

Similar equations for the electric field can be obtained by duality. In the original 12 split equations, σ_z is associated with the d/dz portion of the curl operator. Here σ_z can be associated with a mapping for the z variable into complex space [7], [8]. Rearrangement of (3a) results in

$$(j\omega\mu_0 + \sigma_z^*)\tilde{H}_x = jk_y\tilde{E}_z + jk_y\frac{\sigma_z^*}{j\omega\mu_0}\tilde{E}_z - jk_z\tilde{E}_y. \tag{4}$$

Multiplication by jw results in a second-order time derivative for H_x in the time domain. Implementation of this scheme results in a less efficient algorithm then the original split-field PML. An alternate approach is to define an intermediate variable to absorb the effect of the extra jw term. Two methods are presented that have identical storage requirements, but different CPU requirements per iteration. Transformation of (4) into the time domain produces a modified Maxwell's equation with an additional source term, denoted by F_z, that is dependent upon the integral of the normal component of the electric field

$$\left(\mu_0\frac{\partial}{\partial t} + \sigma_z^*\right)H_x = \frac{\partial E_y}{\partial x} - \frac{\partial E_z}{\partial y} - \frac{\partial F_z}{\partial y} \tag{5}$$

where

$$F_z = \frac{\sigma_z^*}{\mu_0}\int_0^t E_z\, dt. \tag{6}$$

This method provides physical insight into the action of the PML medium in terms of dependent sources. However, it is not the most convenient formulation for numerical implementation. Alternatively, this source F_z can be incorporated with E_z into the variable P_z

$$\left(\mu_0\frac{\partial}{\partial t} + \sigma_z^*\right)H_x = \frac{\partial E_y}{\partial z} - \frac{\partial P_z}{\partial y} \tag{7}$$

where P_z is given by

$$\frac{\partial}{\partial t}P_z = \left(\frac{\partial}{\partial t} + \frac{\sigma_z^*}{\mu_0}\right)E_z. \tag{8}$$

Similar equations can be derived for the other field components tangential to the direction of propagation, i.e. H_y, E_x, and E_y. The electric field equations are modified by the dual variables G_z and Q_z for F_z and P_z, respectively. The equations for the normal components, E_z and H_z, are unchanged.

The six field equations are discretized using the Yee leapfrog scheme with central differencing, with the following representative equation:

$$H_x^{n+1/2} = \frac{1 - \sigma_z^*\Delta t/2\mu_0}{1 + \sigma_z^*\Delta t/2\mu_0}H_x^{n-1/2} + \frac{\Delta t/\mu_0}{1 + \sigma_z^*\Delta t/2\mu_0} \cdot \left[\frac{\partial}{\partial z}E_y^n - \frac{\partial}{2y}(E_z^n + F_z^n)\right] \tag{9}$$

where spatial indexes have been removed for notational clarity. The auxiliary integrals are performed using a running sum trapezoid rule integration method

$$F_z^n = \frac{\sigma_z^*}{\mu_0}\sum_{m=0}^{n-1} E_z^{m+1/2}\Delta t = F_z^{n-1} + \frac{\sigma_z^*\Delta t}{2\mu_0}(E_z^{n-1} + E_z^n). \tag{10}$$

The update of F_z^n is performed in two steps to avoid the back-storing of any variables. The contribution from E_z^{n-1} is added to F_z^{n-1}. Then E_z^n is computed from $E_z^{n-1}, H_y^{n-1/2}, H_z^{n-1/2}$, and $G_z^{n-1/2}$. In the edge and corner PML regions the split equations are updated. On the interfaces, the split tangential electric field components are combined into total field components. Finally, F_z^n is completed by adding the contribution from E_z^n. A similar two-step process is done for $G_z^{n+1/2}$. For the second method, discretization of (8) yields

$$P_z^n = P_z^{n-1} + \left(\frac{\sigma_z^*\Delta t}{2\mu_0} - 1\right)E_z^{n-1} + \left(\frac{\sigma_z^*\Delta t}{2\mu_0} + 1\right)E_z^n. \tag{11}$$

Again a two-step process is used to calculate P_z^n efficiently.

The CPU and memory savings of the modified PML formulation over the split field formulation can be demonstrated by considering a free space radiation problem within a cubical domain N cells on a side surrounded by a PML medium of D layers. The FDTD implementation of the original PML equations results in the storage requirements for 12 $N \times N \times D$ arrays for the split field components in the six wall regions, requiring a total of 48 additions and 16 multiplications per iteration. For the special case of a wall PML region with propagation in the z-direction, E_z and H_z do not need to be split. This 10-unknown PML region requires 40 additions and 14 multiplications. Both unsplit formulations require only 8 $N \times N \times D$ arrays in the six wall regions. Fig. 2 shows that memory savings of up to 20% can be achieved within the PML as the ratio of Yee size to PML depth is varied. The dependent source implementation results in 36 additions and 14 multiplications for cubic cells. For a PML depth to Yee dimension ratio of 0.1, roughly 7% CPU time savings within the PML is attained. The second implementation eliminates eight additions per iteration resulting in up to 22% time savings within the PML (14% for general noncubical cells). As compared to the original 12 unknown formulation, savings in memory and runtime can be as much as 33%. Both unsplit formulations provide the same memory benefits; however, the second implementation not only performs more efficiently but is also more attractive for parallel implementation.

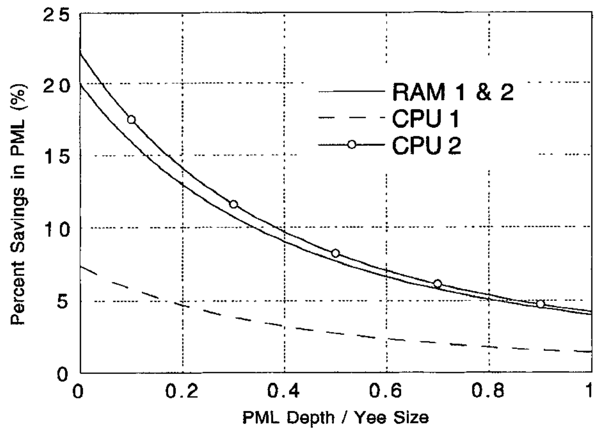

Fig. 2. Percent reduction in memory and runtime requirements within PML regions as ratio of number of PML layers to number of cubic Yee cells along an edge of a cubical volume is varied.

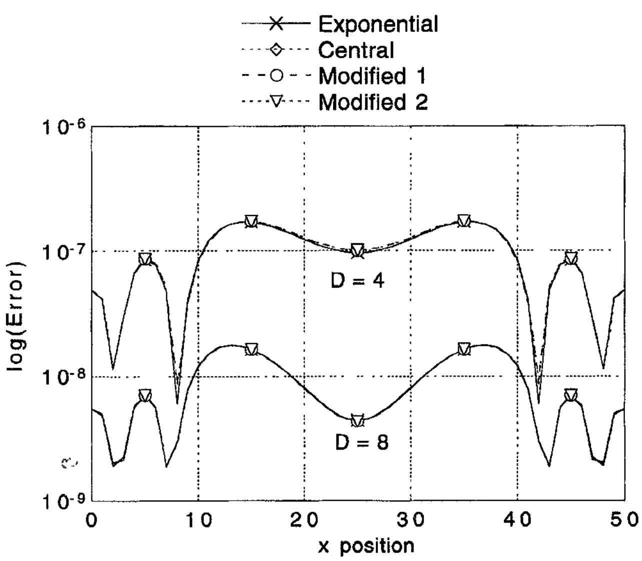

Fig. 3. Local error observed at $E_z(x,0,0)$ at time step 100 for various PML implementations. Parabolic profile with $R = .001\%; \Delta x = 15$ mm, $\Delta t = 25$ ps.

IV. Numerical Examples

Several test cases are investigated to ascertain the accuracy levels that are associated with the unsplit PML formulation. First, a simple point source radiating in free space is considered. PML is used to truncate the computational volume of the test region to $50 \times 50 \times 51$ cubical cells. The local error caused by reflections from the PML is computed by subtracting the test field from the corresponding space-time field in a much larger reference domain. E_z at the center of the domain is excited with a smooth compact pulse [9]. Fig. 3 shows the absolute error at time step 100 along the line $(x, 0, 0)$ for various PML implementations. The difference between exponential and central differencing is seen to be insignificant. The results for the reduced unknown formulations are indistinguishable from the split formulation results.

To test the effectiveness of the modified formulation without the aid of the corner regions, a rectangular X-band waveguide is studied. A 10.0-GHz ramped sinusoidal excitation [10] is used to operate above the 6.557-GHz cutoff frequency of the dominant TE_{10} mode. The reflection error due to the PML termination is calculated at the center of the termination plane. Reflections of -71.3 dB are obtained with the split field formulation as well as both modified formulations for a 12-layer PML with fourth-order conductivity profiles. However, the modified PML is more sensitive to the order of the conductivity profile than the regular PML suggesting that the modified formulations are more prone to reflections caused by the larger discontinuities in σ that occur in the first few layers with low-order profiles. Because higher-order profiles do not increase the computational demand, the modified PML efficiently provides accuracy levels comparable to the more expensive split PML.

V. Conclusion

An efficient reduced field implementation of the Berenger perfectly matched layer concept has been presented. By combining the unsplit formulation in the wall regions with the split formulation in the edges and corners, memory and CPU time requirements may be reduced by up to 33% when compared to the original 12-unknown formulation or 20% when compared to the 10-unknown formulation specialized for wall regions. The reduced method cannot be applied in the edge and corner regions because the overlapping conductivities make the expressions for the tangential components less efficient than the split formulation. Research is currently underway on removing this restriction.

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